Cauchy-Goursat Theorem

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44. Cauchy-Goursat theorem

Cauchy theorem: Let C be a simple closed contour described in the positive sense. Let f be analytic at each point interior to and on C. Let f^{1} be continuous in the closed region R consisting of all points interior to and on the simple closed contour C. then $\int_c f(z)dz = 0$.

OR

If a function f is analytic and f^{1} is continuous at all points interior to and on a simple closed contour C, then $\int_c f(z)dz = 0$.

Proof:

We let C denote a simple closed contour z = z(t), $(a \leq t \leq b)$ described in the positive sense, and f is analytic and f^1 is continuous at all points interior to and on C.

Now, $\int_c f(z)dz = \int_a^b f[z(t)]z^1(t)dt.$ ....(1)

i.e., $\int_c f(z)dz = \int_a^b (u[x(t), y(t)] + iv[x(t), y(t)])(x^1(t) + iy^1(t)dt$

i.e. $\int_c f(z)dz = \int_a^b (ux^1 - vy^1)dt + i\int_a^b (vx^1 + uy^1)dt.$.....(2)

i.e., $\int_c f(z)\,dz = \int_c (udx - vdy) + i\int_c (vdx + udy)$ ....(3)

Formula from calculus:

Suppose that two real-valued functions P(x, y) and Q(x, y), together with their first-order partial derivatives, are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C. According to Green's theorem, $\int_c Pdx + Qdy = \iint_R (Q_x - P_y)\,dA$

Now f is analytic in R $\Rightarrow$ f is continuous in R

Then the function u and v are also continuous in R.

If f^1 is continuous in R then so are the first-order partial derivatives of u and v.

Then from Green's theorem we can rewrite (3) as

$$\int_c f(z)dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA$$

i.e., $\int_c f(z)dz = 0$ (since $u_x = v_y$, $u_y = -v_x$)

Remark: Suppose C is taken in the clockwise direction then

$$\int_c f(z)\,dz = -\int_{-c} f(z)\,dz = 0$$

Example: If C is any simple closed contour, in either direction, then $\int_c \exp(z^3)\,dz = 0$. We know that f(z) = exp(z^3) is analytic everywhere and its derivative $f^1(z) = 3z^2\exp(z^3)$ is continuous everywhere.

Hence, $\int_c \exp(z^3)\,dz = 0$.

45. Proof of Cauchy-Goursat theorem

Statemen:(Cauchy-Goursat) If a function f is analytic at all points interior to and on a simple closed contour C, then $\int_{c} f(z)\,dz = 0$

Proof of the theorem

We first prove the following Lemma.

Lemma: Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself.

For any positive number ε, the region R can be covered with a finite number of squares and partial squares, indexed by j = 1, 2, ..., n, such that in each one there is a fixed point z_j for which the inequality

$$\left|\frac{f(z)-f(z_j)}{z-z_j}-f^1(z_j)\right|<\varepsilon \quad (z\neq z_j)........(1)$$

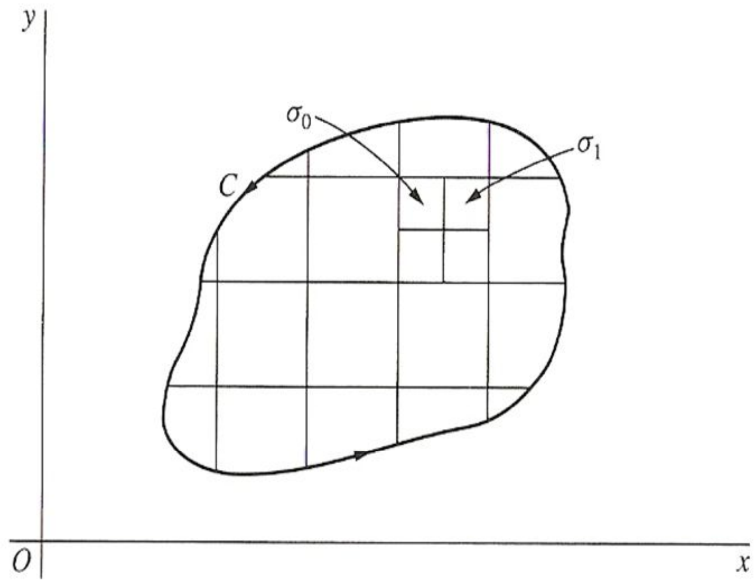
is satisfied by all other points in that square or partial square.

Proof of the Lemma: We start by forming subsets of the region R which consists of the points on a positively oriented simple closed contour C together with points interior to C.

We draw equally spaced lines parallel to the real and imaginary axes such that the distance between adjacent vertical lines is the same as that between adjacent horizontal lines.

We thus form a finite number of closed square sub regions, where each point of R lies in at least one square sub regions (square), where each point of R lies in at least one such square or partial Square (if a particular square contains points that are not in R, we remove those points and call what remains a partial square) and each square or partial square contains points of R.

We thus cover the region R with a finite number of squares and partial squares.



FIG

We suppose that the needed points z_j do not exist after subdividing one of the original sub regions a finite number of times and reach a contradiction. We let σ_0 denote that sub regions if it is a square; if it is a partial squares, we let σ_0 denote the entire square of which it is a part. After we subdivide σ_0, at least one of the four smaller squares, denoted by σ_1, must contain points of R but no appropriate point z_j. We then subdivide σ_1 and continue in this manner. It may be that after a square σ_{k-1} (k = 1, 2, ...,) has been subdivided, more than one of the four smaller squares constructed from it can be chosen. To make a specific choice, we take σ_k to be the one lowest and then furthest to the left.

We construct the nested infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k, \ldots$ (2), of squares such that there is a point z_0 common to each σ_k ; also, each of these squares contains points of R other than possibly z_0.Recall how the sizes of the squares in the sequence are decreasing, and note that any δ neighborhood $|z - z_0| < \delta$ of z_0 contains such squares when their diagonals have lengths less than δ .

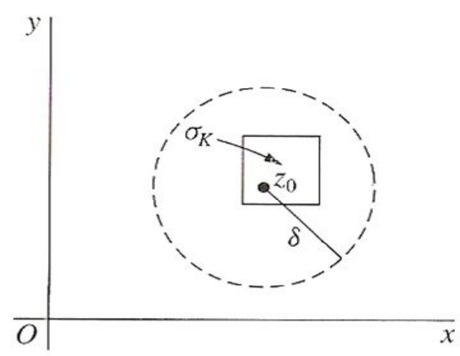


FIGURE 55

Every δ neighborhood $|z-z_0|<\delta$ therefore contains points of R distinct form z_0, and this means that z_0 is an accumulation point of R. Since the region is a closed set, it follows, that z_0 is a point in R.

Given: f is analytic in R $\Rightarrow$ it is analytic at z_0.

So $f^1(z_0)$ exists.

i.e., for each $\varepsilon>0, \exists\ |z-z_0|<\delta$ such that $\left|\dfrac{f(z)-f(z_0)}{z-z_0}-f^1(z_0)\right|<\varepsilon$

But the neighborhood $|z - z_0| < \delta$ contains a square σ_k when the integer k is large enough that the length of a diagonal of that square is less then δ. Consequently, z_0 serves at the point z_j in inequality (1) for the sub region consisting of the square σ_k or a part of σ_k. Contrary to the way in which the sequence (2) was formed, then, it is not necessary to subdivide σ_k. We then arrive at a contradiction, and the proof of the lemma is complete.

Proof of Cauchy-Goursat theorem:

<u>To Prove:</u> $\int_c f(z)\,dz=0$ ...(3) Where f is analytic through a region R consisting of a positively oriented simple closed contour C and points interior to it.

Given $\varepsilon>0$, we consider the covering of R into a finite number of squares and partial squares. Let us define on the jth square or partial square the following function, where z_j is the fixed point in that sub region for which inequality (1) holds:

$$\delta_j(z)=\begin{cases}\dfrac{f(z)-f(z_j)}{z-z_j}-f^1(z_j) & when \quad z\neq z_j \\ 0 & when \quad z=z_j\end{cases} \quad ....(4)$$

From (1), $|\delta_j(z)| < \varepsilon$ .....(5) at all points z in the sub region of which $\delta_j(z)$ is defined. Since f(z) is continuous, $\delta_j(z)$ is continuous throughout the sub region and $\lim_{z \to z_j} \delta_j(z) = f^1(z_j) - f^1(z_j) = 0$

Next , let C_j (j = 1, 2, ..., n) denote the positively oriented boundaries of the of the above squares or partial squares covering R. Let z be a point on any particular C_j. Then from (4),

$f(z) - f(z_j) - (z - z_j) f^1(z_j) = (z - z_j)\delta_j(z)$

i.e., $f(z) = f(z_j) + (z - z_j) f^1(z_j) + (z - z_j)\delta_j(z)$

i.e., $f(z) = f(z_j) - z_j f^1(z_j) + f^1(z_j) z + (z - z_j)\delta_j(z)$

i.e., $\int_{c_j} f(z)\,dz = \left[f(z_j) - z_j f^1(z_j)\right]\int_{c_j} dz + f^1(z_j)\int_{c_j} z\,dz + \int_{c_j} (z - z_j)\,\delta_j(z)\,dz$ ……(6)

i.e., $\int_{c_j} f(z)\,dz = \left[\int_{c_j} (z - z_j)\,\delta_j(z)\,dz\right]$ (j =1, 2, …, n) ……..(7)

since $\int_{c_j} dz = 0$ and $\int_{c_j} z\,dz = 0$ as the functions 1 and z possess anti - derivatives everywhere in the finite plane.

Then $\sum_{j=1}^{n}\int_{c_j} f(z)\,dz = \sum_{j=1}^{n}\int_{c_j} (z - z_j)\,\delta_j(z)\,dz$

i.e., $\int_{c} f(z)\,dz = \sum_{j=1}^{n} \int_{c_j} (z - z_j)\,\delta_j(z)\,dz$

since the two integrals along the common boundary of every pair of adjacent sub regions cancel each other, the integral being taken in one sense along that line segment in one sub region and in the opposite sense in the other (Fig). Only the integrals along the arcs that arc parts of C remain.

(8) $$\left|\int_C f(z)\,dz\right| \le \sum_{j=1}^{n}\left|\int_{C_j}(z-z_j)\delta_j(z)\,dz\right|.$$

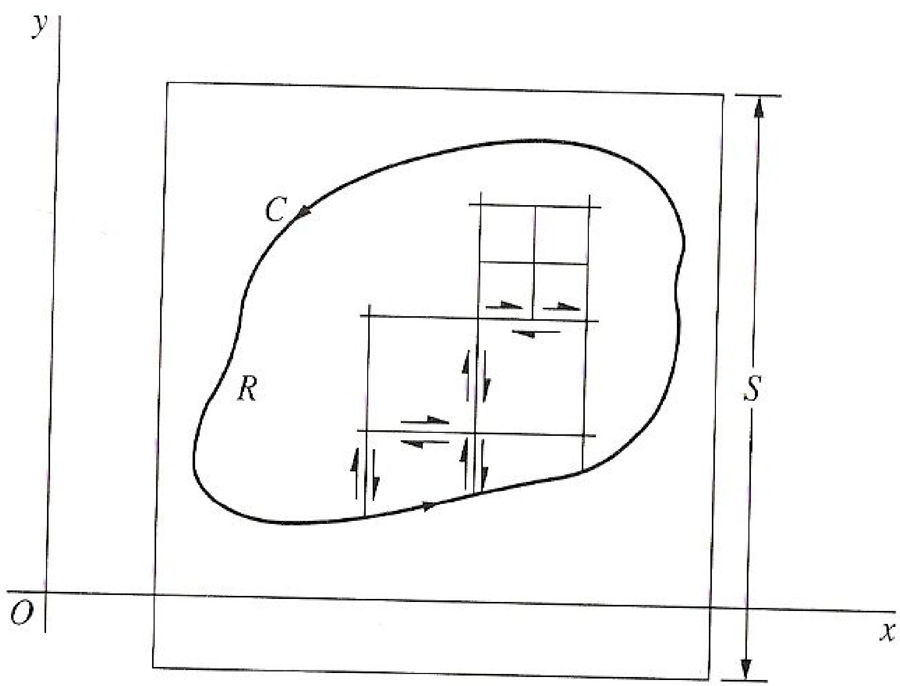


FIGURE 56

So $\left|\int_c f(z)dz\right| \leq \sum_{j=1}^{n}\left|\int_{c_j}(z-z_j)\delta_j(z)dz\right| \ldots..(8)$

To find an upper bound for each absolute value on the right in (8).

Note that each C_j coincides either entirely or partially with the boundary of a square. In either case, we let s_j denote the length of side of the square. In the jth integral, both z and z_j lie in C_j and so $\left|(z-z_j)\delta_j(z)\right| < \sqrt{2}s_j\varepsilon \ldots..(9)$

Note that the length of C_j is 4 s_j if C_j is the boundary of a square. Let A_j be the area of the square. So $\left| \int_{c_j} (z-z_j)\delta_j(z)\,dz \right| < \sqrt{2}\, s_j \varepsilon\, 4s_j = 4\sqrt{2}\, A_j \varepsilon$ ….(10)

If C_j is the boundary of a partial square, its length does not exceed 4 s_j + L_j Where L_j is the length of that part of C_j which is also a part of C, Again letting A_j denote the area of the full square,

we find that

$$\left| \int_{c_j} (z-z_j)\delta_j(z)\,dz \right| < \sqrt{2}\, s_j \varepsilon (4s_j + L_j)$$

i.e., $\left|\int_{c_j}(z-z_j)\delta_j(z)\,dz\right| < 4\sqrt{2}\,A_j\,\varepsilon + \sqrt{2}\,S\,L_j\,\varepsilon$ ....(11)

where S is the length of a side of some square that encloses the entire contour C as well as all of the squares originally used in covering R. Note that the sum of all the A_j's does not exceed S^2.

If L denotes the length of C, it follows from (8), (10), and (11) that

$$\left|\int_{c} f(z)\,dz\right| < (4\sqrt{2}\,S^2 + \sqrt{2}\,S\,L)\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, (we can choose it so that the right hand side of this last inequality is as small as we please. The left-hand side, which is independent of ε, must therefore be equal to; hence) $\left|\int_{c} f(z)\,dz\right| = 0$.